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THEORETICAL ARTICLE

Duality in the fuzzy-parametric space for fuzzy-parametric nonlinear programming problem

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Abstract

The concepts of fuzziness and parametric analysis are of importance to treat uncertainty in mathematical model and may offer certain more viewpoints. The basic notions of the parametric study in nonlinear programming problem are presented by Osman (Aplikace matematiky 22(5):318–332, 1977; Aplikace matematiky 22(5):333–348, 1977). In general, a parametric programming problem is not easy to be solved. In addition, sometime, solving a parametric programming problem with parameters in the objective is easier than solving a parametric problem with parameters in the constraints and vice versa. Therefore, a parametric study in duality space is important to facilitate solving a parametric programming problem. The fuzzy nonlinear problem is interested area for research as one of the tools for treating uncertainty. The fuzzy nonlinear problem when parameters in the objective function or constrains or both is called the fuzzy parametric nonlinear problem. Therefore, dealing with fuzziness and duality parametric space is concerned. In this paper, a novelty introduction to the fuzzy basic notions of parametric programming problem are clarified, the relations between the concepts concerning duality in parametric spaces which introduced by Osman et al. (Int J Math Arch 6(12):161–165, 2016) and the fuzzy concepts are presented. We present and define the fuzzy parametric notions of the set of feasible parameters, the solvability set, and the stability sets of the first and second kind. These notions are not defined before. The theoretical relations and an illustration example are introduced.

Keywords Nonlinear programming problem · Parametric · Duality · Fuzzy set · Fuzzy nonlinear programming problem

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1 Introduction

In general, the dual problem leads to specialized algorithms for some important classes of programming problem. For instance, in linear programming problems, solving transportation simplex method and the network simplex method rely partly on duality. In addition, sometimes the dual is just easier to be solved. Moreover, the dual problem gives a better understanding of the solution and may offer certain more perspectives.

Some or all of the mathematical model inputs are subject to sources of uncertainty, including errors of measurement, absence of information and poor or partial understanding of the driving forces and mechanisms. In such case, one of the trusted techniques is to use a parametric analysis for the problem which may be helpful for better understanding the solution. Another technique for treating uncertainty is done by using the concept of fuzziness. Therefore, dealing with fuzziness and duality in parametric space is concerned.

When addressing real world problems, frequently the parameters are imprecise numerical quantities. Fuzzy quantities are very adequate for modeling these situations [10]. Bellman and Zadeh [2] introduced the concept of fuzzy quantities and the concept of fuzzy decision making. For making a comparison among fuzzy alternatives, it is useful to convert fuzzy numbers into crisp numbers. Therefore, the most common approach to solve a fuzzy nonlinear programming (NLP) problem is to change it into the corresponding crisp NLP problem.

1.1 Fuzzy set

A fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x)\right) | x \in X \right\}.$$

Here $\mu_{\tilde{A}}(x)$ is interpreted as the membership function and the values of $\mu_{\tilde{A}}(x)$ at x represent the "grade of membership" of $x \in X$ in $\tilde{A}$. The membership function $\mu_{\tilde{A}}(x)$ associates with each point $x \in X$ a real number in the interval [0, 1].

1.2 Fuzzy number

A fuzzy number $\tilde{A}$ is a fuzzy set defined on $\mathbb{R}$ (membership function $\mu_{\tilde{A}}(x)$: $\mathbb{R} \rightarrow [0, 1]$), whose membership function satisfies the following properties

(1) $\tilde{A}$ is normal i.e. there exist an element $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}\left(x_0\right) = 1$.
(2) $\tilde{A}$ is convex i.e.
$\mu_{\tilde{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left\{\mu_{\tilde{A}}\left(x_1\right), \mu_{\tilde{A}}\left(x_2\right)\right\}; x_1,\ x_2 \in \mathbb{R},\quad \forall \lambda \in [0, 1].$
(3) $\mu_{\tilde{A}}(x)$ is upper continuous.
(4) supp $(\tilde{A})$ is bounded, where supp$(\tilde{A}) = \left\{x \in X \middle| \mu_{\tilde{A}}(x) > 0\right\}$.

1.3 Generalized fuzzy number

A generalized fuzzy number $\tilde{A}$ is a fuzzy set defined on $\mathbb{R}$ whose membership function satisfies the following properties

(1) $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to [0, 1].
(2) $\mu_{\tilde{A}}(x) = 0, \ -\infty < x \leq a$.
(3) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a, b]$.
(4) $\mu_{\tilde{A}}(x) = 1, \ b \leq x \leq c$.
(5) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $[c, d]$.
(6) $\mu_{\tilde{A}}(x) = 0, \ d \leq x < \infty$.

where a, b, c, d are real numbers.

1.4 Trapezoidal fuzzy number

A fuzzy set $\tilde{A}$ defined on $\mathbb{R}$ is called trapezoidal fuzzy number (as shown in Fig. 1) and is denoted by $\tilde{A}=(a, b, c, d)$ if the membership function of $\tilde{A}$ is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{a-x}{a-b}, & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \frac{d-x}{d-c}, & c \leq x \leq d, \\ 0, & otherwise \end{cases}, \quad a \leq b \leq c \leq d$$

Note If b and c are equal, then the trapezoidal fuzzy number becomes a triangular fuzzy number as shown in Fig. 2 and is denoted as $\tilde{A}=(a, b, d)$.

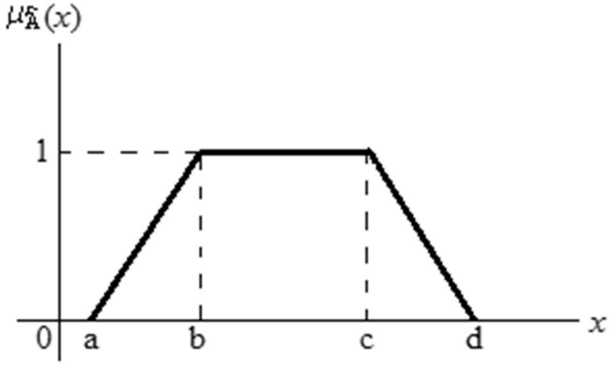


Fig. 1 Trapezoidal membership function

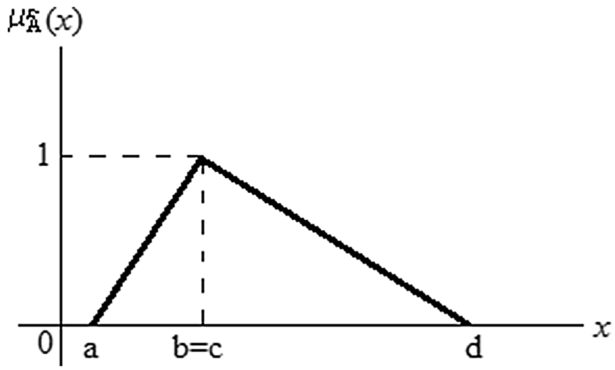


Fig. 2 Triangle membership function

1.5 Defuzzification

Defuzzification is a procedure of transforming fuzzy values to crisp values. Defuzzification method provides a correspondence from the set of all fuzzy sets into the set of all real numbers.

1.6 Parametric notions

Osman [4, 14, 15] introduced the notions of the set of feasible parameters, the solvability set, and the stability sets of the first and second kinds and analyzed these concepts for parametric convex programming problems. Moreover, the new concept of duality in parametric space that introduced in [16], In general, represents a new vision of getting the dual of parametric problem by utilizing the direct and clear relations between different scalarization approaches that used for treating multiobjective nonlinear programming (MONLP) problem.

1.7 Multiobjective optimization

Mathematically, the multiobjective optimization problem can be formulated as a vector optimization programming problem under certain constraints [20]. One of the well-known techniques for solving the multiobjective programming problem, the ε-constraint method is a procedure which overcomes some of the convexity problems of the weighted sum technique [1]. It involves minimizing (in case of minimization) a primary objective and expressing the other objectives as constraints. This method can identify all efficient solutions on a nonconvex boundary, which are not obtainable using the weighted sum approach [10].

When several objective functions are present, the optimal solution for an objective function may not be an optimal solution for some other objective functions. Therefore, one needs to use the concept of the "best compromise solution", also known as “non- dominated solution”, “efficient solution”, “Pareto optimal solution”, “Pareto efficient solution” etc. Thus, in multiobjective programming problem, the notion of efficiency is introduced to replace with that of optimality. A solution is called efficient if none of the objective functions can be developed in value without demeaning the value of any other objective [6–9, 17].

Furthermore, there is no absolute and fixed best solution to the multiobjective optimization programming in the real world since everything is relative in nature. Thus, the concept of the best solution is another relative characteristic from one to another [3, 13, 18, 19]. Thus, studies and researches have been done by researchers and scholars to formulate the original basic problem in the multiobjective optimization model. They found various methods and approaches for this purpose and optimal solutions relative to them. On the other hand they looked for a Pareto solution (efficient solution) (see, [6–9, 11, 12, 17]).

Thus, in this paper, we are interested in introducing a powerful tool that is capable for representing most of real life situations (problems). Therefore, novelty basic fuzzy parametric notions are introduced in a duality in the fuzzy-parametric space for fuzzy-parametric NLP problem. This paper is organized as follows: In Sect. 2, the problem definition is introduced. In Sect. 3, Theoretical relations between the two problems (primal and the dual one) are illustrated. Section 4, an illustrative example is introduced. Section 5, conclusion is presented.

2 Problem definition

Consider the following two fuzzy-parametric nonlinear programming problems where the first one ($\tilde{P}_1(\lambda, \tilde{A})$) with parameters ($\lambda$) in the objective function and the second ($\tilde{P}_2(\epsilon, \tilde{A})$) with parameters ($\epsilon$) in the constraints which are formulated as follows:

$$\tilde{P}_1(\lambda, \tilde{A}):\quad Min \sum_{i=1}^{k+1} \lambda_i f_i(x, \tilde{a}_i) \quad Subject\ to \quad \tilde{M} = \left\{ x \in R^n \,\middle|\, \begin{array}{l} g_r(x, \tilde{b}_r) \le 0, \quad r = 1, 2, \dots, m, \\ \sum_{i=1}^{k+1} \lambda_i = 1, \quad \lambda_i \ge 0 \end{array} \right\}$$

$$\tilde{P}_2(\epsilon, \tilde{A}):\quad Min\, f_1(x, \tilde{a}_1), \quad subject\ to \quad \tilde{N} = \left\{ x \in R^n \,\middle|\, \begin{array}{l} f_i(x, \tilde{a}_i) \le \epsilon_i, \quad i = 2, \dots, (k+1), \\ g_r(x, \tilde{b}_r) \le 0, \quad r = 1, 2, \dots, m \end{array} \right\}$$

where:

- $\lambda_i, \ \epsilon_i \in R, \forall\, i.$
- $\tilde{A} = (\tilde{a}_i, \tilde{b}_r)$ is a vector of fuzzy numbers and $\mu_{\tilde{A}}$ represents the membership functions.

When using the same defuzzification method, problems $\tilde{P}_1(\lambda, \tilde{A})$ and $\tilde{P}_2(\epsilon, \tilde{A})$ are said to be fuzzy parametrically dual. For example, Dutta et al. [5] shown that α-cut method is a method general enough to deal with all kinds of fuzzy arithmetic including nth root, exponentiation and taking log. Thus, for instance, α-cut approach will be used for defuzzification problems $\tilde{P}_1(\lambda, \tilde{A})$ and $\tilde{P}_2(\epsilon, \tilde{A})$ for transformed them into the following crisp forms respectively.

$$P_1(\lambda):\quad Min \sum_{i=1}^{k+1} \lambda_i f_i(x, a_i) \quad Subject\ to \quad M = \left\{ X \in \Omega \,\middle|\, \begin{array}{l} g_r(x, b_r) \le 0,\ r = 1, 2, \dots, m, \\ \mu_{\tilde{A}} \ge \alpha,\ 0 \le \alpha \le 1,\ \sum_{i=1}^{k+1} \lambda_i = 1 \\ \lambda_i \ge 0 \end{array} \right\}$$

$$P_2(\epsilon):\quad Min f_1(x, a_1), \quad subject\ to \quad N = \left\{ X \in \Omega \,\middle|\, \begin{array}{l} f_i(x, a_i) \le \epsilon_i,\ i = 2, \dots, (k+1), \\ g_r(x, b_r) \le 0,\ r = 1, 2, \dots, m \\ \mu_{\tilde{A}} \ge \alpha,\ 0 < \alpha \le 1 \end{array} \right\}$$

where $X = (x, a_i, b_r) \in \Omega = R^n \times R^{k+1} \times R^m$. Problems $P_1(\lambda)$ and $P_2(\epsilon)$ are said to be parametrically dual. For illustration, at the next section, some definitions and theorems are presented.

2.1 Basic notions

Let O_1, O_2 denote the sets of optimal solutions for problems $P_1(\lambda)$, $P_2(\epsilon)$ respectively. In addition, let $\bar{X} \in O_1$ and $X^* \in O_2$. Then, for problem $P_1(\lambda)$, the following basic notions are defined:

(1) The solvability set B_1:

$$B_1 = \left\{ \lambda \in R^{k+1} \middle| O_1 \ne \phi \right\}$$

(2) The fuzzy stability set of the first kind $S_1(\bar{X})$:

$$S_1(\bar{X}) = \left\{ \lambda \in B_1 \middle| \bar{X} \in O_1 \right\}$$

(3) The fuzzy stability set of the second kind $Q_1(\sigma(I))$:

$$Q_1(\sigma(I)) = \left\{ \lambda \in B_1 \middle| \exists X \in O_1 \cap \sigma(I) \right\} :$$

where $\sigma(I)$ is defined as:

$$\sigma(I) = \left\{ X \in M \middle| g_\gamma(x, b_\gamma) = 0,\ \gamma \subset \{1, 2, \cdots, m\} \right\}.$$

For problem $P_2(\epsilon)$, the following basic notions are defined:

(1) The set of feasible parameters F:

$$F = \left\{ \epsilon \in R^k | N \ne \phi \right\}$$

(2) The solvability set B_2:

$$B_2 = \{\epsilon \in F | O_2 \neq \phi\}$$

(3) The fuzzy stability set of the first kind $S_2(\bar{X})$:

$$S_2(\bar{X}) = \{\epsilon \in B_2 | \bar{X} \in O_2\}$$

(4) The fuzzy stability set of the second kind $Q_2(\sigma(I))$:

$$Q_2(\sigma(I)) = \{\epsilon \in B_2 | \exists X \in O_2 \cap \sigma(I)\}$$

The sets B_1, S_1, Q_1 are said to be parametrically dual to the sets B_2, S_2, Q_2 respectively under the convexity conditions of the set M and the function $f_i(x, a_i)$, $\forall\, i$. Otherwise (in the nonconvex case), they are said to be partially parametrically dual.

3 Theoretical relations

Let us consider the following fuzzy multiobjective nonlinear programming problem $(\tilde{P}_0)$ and its corresponding crisp nonlinear programming problem (P_0), by using ∝-cut approach.

$\tilde{P}_0$:

$$Min\, f_i(x, \tilde{a}_i),\quad i = 1, 2, \dots, k+1$$

Subject to

$$\tilde{M}_0 = \{x \in R^n | g_r(x, \tilde{b}_r) \le 0,\quad r = 1, 2, \dots, m,\}$$

P_0 :

$$Min\ f_i(x,\ a_i),\quad i = 1,\ 2, \dots,\ k+1$$

Subject to

$$M_o = \left\{ X \in \Omega \,\middle|\, \begin{array}{l} g_r(x,\ b_r) \le 0,\ r = 1,\ 2, \dots,\ m, \\ \mu_{\tilde{A}} \ge \alpha,\ 0 \le \alpha \le 1 \end{array} \right\}$$

It is well known that the efficient solution of problem P_0 could be generated by using either one of the following scalarization problems $P_1(\lambda)$ and $P_2(\epsilon)$. Where $P_1(\lambda)$ represents the weighted problem that associated with the weighted method and $P_2(\epsilon)$ represents the epsilon-constraint problem that associated with the epsilon-constraint method. The relations between the efficient solutions of problem P_0 and the optimal solutions of problems $P_1(\lambda)$ and $P_2(\epsilon)$ are given through the following theorems. But first, an illustration of the relations between the solution given by the weighted method and the epsilon-constraint method are introduced.

The solution of the weighting method is always Pareto optimal if the weighting coefficients are all positive or if the solution is unique, without any further assumptions. The weakness of the weighting method is that not all of the Pareto optimal solutions can be found unless the problem is convex. Therefore, Any Pareto optimal solution of a convex multiobjective optimization problem can be found by the weighting method. On the other hand, it is possible to find every Pareto optimal solution of any multiobjective optimization problem by the epsilon-constraint method (regardless of the convexity of the problem) [12].

Therefore, a Pareto optimal solution of a convex multiobjective can be found by one of the two methods (weighted or epsilon-constraint) and the solution will be optimal for the two problems that associated with each method. While under non-convexity the epsilon-constraint has the ability to get all Pareto solutions. Thus, under convexity of multiobjective problem, an optimal solution of the weighted problem at certain λ can be defined by the epsilon-constraint problem at a corresponding ϵ.

Theorem 1 *If for $\bar{\lambda} \in B_1$, an optimal solution of $P_1(\bar{\lambda})$ is found to be $\bar{X} \in M$. Then $\bar{X}$ is an efficient solution of P_0 if, either $\bar{\lambda} > 0$ or $\bar{X}$ is the corresponding unique optimal solution.*

Theorem 2 *Under convexity assumptions (M is convex and $f_i(x, a_i),\ i = 1, 2, \dots, k + 1$ are convex on M), if $\bar{X}$ is an efficient solution of problem P_0, then there exists $\bar{\lambda} \geq 0$ such that $\bar{X}$ solves problem $P_1(\bar{\lambda})$.*

Theorem 3 *If for $\overline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \in F$, either $\bar{X} \in M$ solves problem $P_2(\overline{\epsilon})$ uniquely or $\bar{X} \in M$ solves all the problems $P_2(\overline{\epsilon})$ for feasible $\overline{\epsilon}$, then $\bar{X}$ is an efficient solution of problem P_0.*

Theorem 4 *If $\bar{X}$ is an efficient solution of problem P_0, then there exists $\overline{\epsilon} \in F$ such that $\bar{X}$ solves problem $P_2(\epsilon)$.*

Utilizing the previous theorems, the following propositions relating the basic notions of problems $P_1(\lambda)$ and $P_2(\epsilon)$ to each other are stated and their proofs could be easily deduced.

Proposition 1 *From the definition of the sets O_1 and O_2, we have $O_1 = O_2$ under the convexity assumptions. Moreover, if the convexity assumptions are not satisfied, we get that $O_1 \subset O_2$.*

Proposition 2 *Under convexity assumptions, If $\lambda \in B_1$, then there exists a corresponding $\epsilon \in B_2$. Also, if $\epsilon \in B_2$ then there exists a corresponding $\lambda \in B_1$.*

Proposition 3 *If $\bar{\lambda} \in S_1$, either $\bar{\lambda} > 0$ or $\bar{X}$ solves problem $P_1(\bar{\lambda})$ uniquely, then there exist a corresponding $\overline{\epsilon} \in S_2$, and if $\overline{\epsilon} \in S_2$ and $\bar{X}$ solves uniquely problem $P_2(\overline{\epsilon})$ then there exists a corresponding $\bar{\lambda} \in S_1$ if the convexity assumptions are satisfied.*

Proposition 4 *If $\bar{\lambda} \in Q_1$, then there exists a corresponding $\overline{\epsilon} \in Q_2$ and if $\overline{\epsilon} \in Q_2$, then there exists a corresponding $\bar{\lambda} \in Q_1$ if the convexity assumptions are satisfied.*

3.1 Reduction of the dimensionality space of problem $P_1(\lambda)$

Utilizing the condition $\sum_{i=1}^{k} \lambda_i = 1$, problem $P_1(\lambda)$ could be formulated in the following equivalent form:

$$\hat{P}_1(\lambda):$$

$$Min\left(f_1\left(x,a_1\right)+\sum_{i=1}^{k+1}\lambda_i\left(f_i\left(x,a_i\right)-f_1\left(x,a_1\right)\right)\right),$$

Subject to

$$\hat{M}=\left\{X\middle|\begin{array}{l} g_r\left(x,b_r\right)\le 0,\quad r=1,2,\ldots,m,\\ \mu_{\tilde{A}}\ge\alpha,\ 0<\alpha\le 1,\ \sum_{i=1}^{k}\lambda_i\le 1,\ \lambda_i\ge 0\end{array}\right\}$$

In this case, the dual parametric notions will be in spaces of the dimension k. It is clear that by using other scalarization methods, several dual parametric problems could be easily derived.

The idea behind presenting the concept of dual fuzzy-parametric problems is to clarify the fruitful relations between the two problems and to discuss the possibility that by solving one of them then the second problem is clearly solved.

3.2 Determining the stability set of the first kind

Consider the problem $P_1(\lambda)$, Kuhn–Tucker stationary conditions can be defined as follows:

$$\nabla_X\left(\sum_i\lambda_i f_i\left(x,a_i\right)+\sum_{j=1}^{m}u_j g_j\left(x,b_j\right)+\sum_{j=(m+1)}^{\omega}u_j\left(\alpha-\mu_{\tilde{A}}\right)\right)=0, \tag{1}$$

$$u_j g_j\left(x,b_j\right)=0,\quad j=1,2,\ldots,m \tag{2}$$

$$u_l\left(\alpha-\mu_{\tilde{A}}\right)=0,\quad l=(m+1),(m+2),\ldots,\omega \tag{3}$$

$$g_j\left(x,b_j\right)\le 0,\quad j=1,2,\ldots,m \tag{4}$$

$$\alpha-\mu_{\tilde{A}}\le 0, \tag{5}$$

$$u_j\ge 0,\quad \forall j \tag{6}$$

By fixing x to the optimal $\bar{x}$, and suppose that system (1) has θ equations and ω multipliers ($u_j,\quad j=1,\ 2,\ \ldots,\ \omega$), system (1) is clearly a linear system of equations which can be written in the following form

$$Q_{\beta=1}\left(\lambda_i,A_\alpha,u_1,u_2,\ldots,u_\omega\right)=0,\quad \beta=1,2,\ldots,\theta,\quad A_\alpha=\left(a_i,b_j\right)$$

While systems (2) and (3) can be written in the following form:

$$H_\eta\left(A_\alpha, u_\eta\right) = 0, \quad \eta = 1, 2, \dots, \omega$$

In addition, systems (4) and (5) can be written in the following form:

$$G_\eta\left(A_\alpha\right) \le 0, \quad \eta = 1, 2, \dots, \omega$$

Thus, at the optimal, Kuhn–Tucker stationary conditions can be written in the following form:

$$Q_{\beta=1}\left(\lambda_i, A_\alpha, u_1, u_2, \dots, u_\omega\right) = 0, \quad \beta = 1, 2, \dots, \theta \tag{7}$$

$$H_\eta\left(A_\alpha, u_\eta\right) = 0, \quad \eta = 1, 2, \dots, \omega \tag{8}$$

$$G_\eta\left(A_\alpha\right) \le 0, \quad \eta = 1, 2, \dots, \omega \tag{9}$$

$$u_j \ge 0, \quad \forall j \tag{10}$$

Then, to satisfy conditions (9) and (10), we have one of the following two cases assuming that system (7) is linearly independent.

Case 1: $\theta \ge \omega$

Thus, without loss of generality, system (7) can be written in the form:

$$\begin{aligned} &u_j = \gamma_j\left(\lambda_i,\ A_\alpha\right), \quad j = 1, 2, \dots, \omega \\ &Q_\beta\left(\lambda_i, A_\alpha, \gamma_1, \gamma_2, \dots, \gamma_\omega\right) = 0, \quad \beta = (\omega+1), (\omega+2), \dots, \theta \end{aligned}$$

Let us define the following sets

$$\begin{aligned} \psi_{1\,\eta} &= \left\{ \lambda_i \middle| \gamma_\eta \ge 0,\ G_\eta\left(A_\alpha\right) \le 0 \right\}, \quad \eta = 1, 2, \dots, \omega, \\ \psi_{2\,\eta} &= \left\{ \lambda_i \middle| \gamma_\eta > 0,\ G_\eta\left(A_\alpha\right) < 0 \right\}, \quad \eta = 1, 2, \dots, \omega, \\ \psi_{3\,\beta} &= \left\{ \lambda_i \middle| Q_\beta\left(\lambda_i, A_\alpha, \gamma_1, \gamma_2, \dots, \gamma_\omega\right) = 0 \right\}, \quad \beta = (\omega+1), (\omega+2), \dots, \theta \end{aligned}$$

Then the stability set of the first find $S(\bar{X})$ of problem $P_1(\lambda)$ can be defined in the following form:

$$S(\overline{X}) = \left(\bigcap_{\eta=1}^{\omega} \psi_{1\,\eta} - \bigcup_{\eta=1}^{\omega} \psi_{2\,\eta} \right) \cap \left(\bigcap_{\beta=(\omega+1)}^{\theta} \psi_{3\,\beta} \right)$$

Case 2 $\theta < \omega$

Thus, without loss of generality, system (7) can be written in the form:

$$u_\beta = \psi_\beta\left(\lambda_i,\ A_\alpha,\ u_{\theta+1}, u_{\theta+2}, \dots, u_\omega\right), \quad \beta = 1, 2, \dots, \theta$$

Let us define the following sets

$$\tau_{1\,\eta} = \left\{ \lambda_i \middle| u_\beta \geq 0,\ G_\beta\left(A_\alpha\right) \leq 0 \right\},\quad \beta = 1, 2, \ldots, \theta,$$
$$\tau_{2\,\eta} = \left\{ \lambda_i \middle| u_\beta \geq 0,\ G_\beta\left(A_\alpha\right) \leq 0 \right\},\quad \beta = (\theta+1), (\theta+2), \ldots, \omega$$
$$\tau_{3\,\eta} = \left\{ \lambda_i \middle| u_\beta > 0,\ G_\beta\left(A_\alpha\right) < 0 \right\},\quad \beta = 1, 2, \ldots, \theta,$$
$$\tau_{4\,\eta} = \left\{ \lambda_i \middle| u_\beta > 0,\ G_\beta\left(A_\alpha\right) < 0 \right\},\quad \beta = (\theta+1), (\theta+2), \ldots, \omega$$

Then the stability set of the first find $S(\bar{X})$ of problem $P_1(\lambda)$ can be defined in the following form:

$$S(\overline{X}) = \left(\bigcap_{\eta=1}^{\theta} \tau_{1\,\eta} \cap \bigcap_{\eta=\theta+1}^{\omega} \tau_{2\,\eta} \right) - \left(\bigcup_{\eta=1}^{\theta} \tau_{3\,\eta} \cup \bigcup_{\eta=\theta+1}^{\omega} \tau_{3\,\eta} \right)$$

By the same method, the stability set of the first find $S(\bar{X})$ of problem $P_{2\,j}(\epsilon)$ can be defined.

4 An illustrative example

For the following dual fuzzy-parametric problems, the fuzzy-parametric space according to the stability sets of the first kind can be illustrated as follows:

$$\begin{array}{ll}
\tilde{P}_1\left(\lambda,\ \tilde{a}_i,\ \tilde{b}_r\right): & \tilde{P}_2\left(\epsilon,\ \tilde{a}_i,\ \tilde{b}_r\right) \\
\quad Min\left(-(1-\lambda)\left(\tilde{a}_{1\,1}x_1\right) - \lambda\left(\tilde{a}_{2\,2}x_2\right)\right), & \quad Min\left(-\tilde{a}_{1\,1}x_1\right), \\
\quad \text{Subject to} & \quad \text{Subject to} \\
\quad x_1^2 + x_2^2 \leq \left(\tilde{b}_{11}\right)^2, & \quad -\tilde{a}_{2\,2}x_2 \leq \epsilon_2, \\
\quad 0 \leq \lambda \leq 1, & \quad x_1^2 + x_2^2 \leq \left(\tilde{b}_{11}\right)^2, \\
\quad x_1,\ x_2 \geq 0, & \quad x_1,\ x_2 \geq 0
\end{array}$$

where all membership functions of the fuzzy numbers ($\tilde{a}_{i\,j}$, $\tilde{b}_{i\,j}\ \forall\ i,\ j$) are represented by triangle shape whose vertices are $\left(t_{1\,j},\ 0\right)$, $\left(t_{2\,j},\ 1\right)$, and $\left(t_{3\,j},\ 0\right)$. Thus, they will have the following form:

$$\mu_{\tilde{a}_{i\,j}} = \begin{cases} \frac{1}{t_{2\,j}-t_{1\,j}}\left(a_{i\,j} - t_{1\,j}\right), & t_{1\,j} \leq a_{i\,j} \leq t_{2\,j} \\ \frac{1}{t_{2\,j}-t_{3\,j}}\left(a_{i\,j} - t_3\right), & t_{2\,j} \leq a_{i\,j} \leq t_{3\,j} \end{cases}$$

Thus, by using α-cut method for defuzzification, we get that:

$$t_{1\,j} + \alpha\left(t_{2\,j} - t_{1\,j}\right) \leq a_{i\,j} \leq t_{3\,j} + \alpha\left(t_{2\,j} - t_{3\,j}\right),\quad 0 < \alpha \leq 1$$

Supposing that the following table represents the required data for all membership functions

	t_{1j}	t_{2j}	t_{3j}
a_{11}	4	5	6
a_{22}	2	3	6
b_{11}	0	1	2

Then by using α-cut approach, we get problems $P_1(\lambda)$ and $P_2(\epsilon)$ as follows:

$$\begin{array}{l} P_1(\lambda): \\ \quad Min\left(-a_{11}x_1(1-\lambda)-a_{22}x_2\lambda\right), \\ \quad \text{Subject to} \\ \quad x_1^2+x_2^2 \le \left(b_{11}\right)^2, \\ \quad \alpha+4 \le a_{11} \le 6-\alpha, \\ \quad \alpha+2 \le a_{22} \le 6-3\alpha, \\ \quad \alpha \le b_{11} \le 3-2\alpha, \\ \quad 0 \le \lambda \le 1,\ 0 \le \alpha \le 1, \\ \quad x_1,\ x_2 \ge 0 \end{array} \qquad \begin{array}{l} P_2(\epsilon): \\ \quad Min\left(-a_{11}x_1\right) \\ \quad \text{Subject to} \\ \quad -a_{22}x_2 \le \epsilon_2, \\ \quad x_1^2+x_2^2 \le \left(b_{11}\right)^2, \\ \quad \alpha+4 \le a_{11} \le 6-\alpha, \\ \quad \alpha+2 \le a_{22} \le 6-3\alpha, \\ \quad \alpha \le b_{11} \le 3-2\alpha, \\ \quad 0 \le \alpha \le 1, \\ \quad x_1,\ x_2 \ge 0 \end{array}$$

It will be easily noted that:

$$F=\left\{\epsilon_2 \in R \,\middle|\, \epsilon_2 \ge -a_{22}\,b_{11},\ \alpha+2 \le a_{22} \le 6-3\alpha,\ \alpha \le b_{11} \le 3-2\alpha,\ 0<\alpha \le 1\right\}$$

Then by using Kuhn–Tucker stationary conditions, the set of optimal solutions of problem $P_1(\lambda)$ and $P_2(\epsilon)$ are defined as follows:

$$O_1=\left\{\bar{X} \in L \,\middle|\, \begin{array}{l} x_1=\frac{a_{11}\,b_{11}(1-\lambda)}{\sqrt{a_{11}^2\,(1-\lambda)^2+a_{22}^2\,\lambda^2}}, \\ x_2=\frac{a_{22}\,b_{11}\lambda}{\sqrt{a_{11}^2\,(1-\lambda)^2+a_{22}^2\,\lambda^2}} \\ \alpha+4 \le a_{11} \le 6-\alpha, \\ \alpha+2 \le a_{22} \le 6-3\alpha, \\ \alpha \le b_{11} \le 3-2\alpha, \\ 0<\alpha \le 1,\ \ 0 \le \lambda \le 1 \end{array}\right\}, \quad O_2=\left\{X^* \in L \,\middle|\, \begin{array}{l} x_1=\sqrt{b_{11}^2-\frac{\epsilon_2^2}{a_{22}^2}}, \\ x_2=\frac{-\epsilon_2}{a_{22}}, \\ \alpha+4 \le a_{11} \le 6-\alpha, \\ \alpha+2 \le a_{22} \le 6-3\alpha, \\ \alpha \le b_{11} \le 3-2\alpha, \\ 0<\alpha \le 1, \end{array}\right\}$$

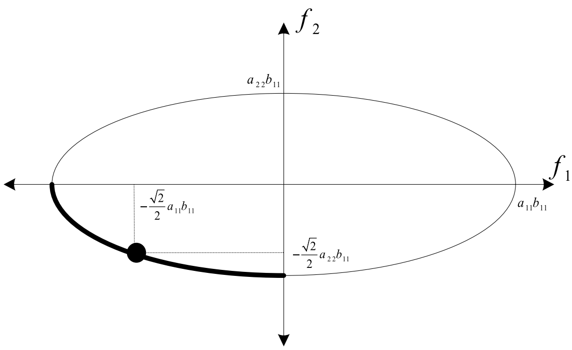
where $L=R^2 \times R^2 \times R$. In addition, from Proposition 1, we get that $O_1=O_2$. Therefore, we get that:

$$\epsilon_2=\frac{-\,a_{22}^2\,b_{11}\,\lambda}{\sqrt{a_{11}^2\,(1-\lambda)^2+a_{22}^2\,\lambda^2}}$$

Moreover, consider for the following multiobjective nonlinear programming problem:

$$\begin{aligned}&\tilde{P}_0 :\\&Min\left(-\tilde{a}_{1\,1}x_1, -\tilde{a}_{2\,2}x_2\right),\\&\text{Subject to}\\&x_1^2 + x_2^2 \le \left(\tilde{b}_{11}\right)^2,\\&x_1,\ x_2 \ge 0\end{aligned}$$

Then by using α-cut approach, we get that:

The problem	Objective space
P_0 : $Min\left(-a_{1\,1}x_1, -a_{2\,2}x_2\right)$ Subject to $x_1^2 + x_2^2 \le \left(b_{11}\right)^2$, $x_1,\ x_2 \ge 0$, $\alpha + 4 \le a_{1\,1} \le 6 - \alpha$, $\alpha + 2 \le a_{2\,2} \le 6 - 3\alpha$, $\alpha \le b_{1\,1} \le 3 - 2\alpha$, $0 < \alpha \le 1$	

As illustrated, $\bar{x} = \left(\frac{\sqrt{2}}{2}b_{1\,1}, \frac{\sqrt{2}}{2}b_{1\,1}\right)$ is an efficient solution for P_0. Thus, from Theorems 2 and 4, $\bar{x}$ is an optimal solution for problem $P_1\left(\bar{\lambda}\right)$ and $P_2\left(\bar{\epsilon}\right)$. Thus, $\bar{X} = \left(\frac{\sqrt{2}}{2}b_{1\,1}, \frac{\sqrt{2}}{2}b_{1\,1},\ a_{1\,1},\ a_{2\,2},\ b_{1\,1}\right) \in O_1 \cap O_2$ and for $\bar{X}$ we get that:

$$S_1\left(\bar{X}\right) = \left\{0 \le \lambda \le 1 \,\middle|\, \lambda = \frac{a_{1\,1}}{a_{1\,1} + a_{2\,2}},\ \bar{X} \in O_1\right\},$$

$$S_2\left(\bar{X}\right) = \left\{\epsilon_2 \in R \,\middle|\, \epsilon_2 = \frac{-\sqrt{2}}{2}\, a_{2\,2}\, b_{1\,1},\ \bar{X} \in O_2\right\}$$

Then for the optimal solution $\bar{X}$ of the two problems $P_1(\lambda)$ and $P_2(\epsilon)$, we get that: $\lambda = \frac{a_{1\,1}}{a_{1\,1}+a_{2\,2}}$ and $\epsilon_2 = \frac{-\sqrt{2}}{2}\, a_{2\,2}\, b_{1\,1}$ are dual parameters. Thus, for the fuzzy optimal solution $\tilde{\bar{X}} = \left(\frac{\sqrt{2}}{2}\tilde{b}_{1\,1}, \frac{\sqrt{2}}{2}\tilde{b}_{1\,1},\ \tilde{a}_{1\,1}, \tilde{a}_{2\,2}, \tilde{b}_{1\,1}\right)$ of problems $\tilde{P}_1\left(\lambda,\ \tilde{A}\right)$ and $\tilde{P}_2\left(\epsilon,\ \tilde{A}\right)$, $\tilde{\lambda} = \frac{\tilde{a}_{1\,1}}{\tilde{a}_{1\,1}+\tilde{a}_{2\,2}}$ and $\tilde{\epsilon}_2 = \frac{-\sqrt{2}}{2}\, \tilde{a}_{2\,2}\, \tilde{b}_{1\,1}$ are fuzzy dual parameters.

5 Conclusion

In this paper, we define the dual fuzzy nonlinear programming problem of a primal fuzzy nonlinear programming problem. In addition, a parametric analysis of the two problems is introduced by defining the fuzzy stability sets. Moreover, a theoretical background is introduced through necessary definitions and different theorems and propositions. The idea behind presenting the concept of fuzzy dual parametric problems is to clarify the fruitful relation between the two problems and to discuss the possibility that by solving one of them, the second problem is clearly solved. Using other scalarization methods will lead to several dual fuzzy parametric problems could be easily derived. So, we will try to apply these concepts and definitions in real applications in the future work. Also, we are interesting to study the properties and definitions of fuzzy-parametric multiobjective linear/nonlinear programming problems.

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